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INITIAL COEFFICIENT BOUNDS FOR SUBCLASSES OF BI-UNIVALENT FUNCTIONS ASSOCIATED WITH THE CHEBYSHEV POLYNOMIALS

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Abstract. In this paper, we introduce and investigate new subclasses of bi-univalent functions defined in the open unit disk, involving special functions associated with Chebyshev Polynomials. Furthermore, we find estimates of first two coefficients of functions in these classes, making use of the Chebyshev polynomials. Also, we give Fekete-Szegö inequalities for these function classes. Several consequences of the results are also pointed out.
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1. Introduction

Let $\mathcal{A}$ denote the class of analytic functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

normalized by the conditions $f(0) = 0 = f'(0) - 1$ defined in the open unit disk

$$\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}.$$

Let $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of functions of the form (1.1) which are also univalent in $\mathbb{U}$. Let $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$ denote the well-known subclasses of $\mathcal{S}$, consisting of starlike and convex functions of order α, $0 \le \alpha < 1$, respectively.

The Koebe one quarter theorem [7] ensures that the image of $\mathbb{U}$ under every univalent function $f \in \mathcal{A}$ contains a disk of radius $\frac{1}{4}$. Thus every univalent function f has an inverse f^{-1} satisfying

$$f^{-1}(f(z)) = z, \ (z \in \mathbb{U}) \text{ and } f(f^{-1}(w)) = w \ (|w| < r_0(f), \ r_0(f) \ge \frac{1}{4}).$$

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A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{U}$ if both f and f^{-1} are univalent in $\mathbb{U}$. Let Σ denote the class of bi-univalent functions defined in the unit disk $\mathbb{U}$. Since $f \in \Sigma$ has the Maclaurian series given by (1.1), a computation shows that its inverse $g = f^{-1}$ has the expansion

$$g(w) = f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 + \cdots . \tag{1.2}$$

An analytic function f is subordinate to an analytic function g, written $f(z) \prec g(z)$, provided there is an analytic function w defined on $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1$ satisfying $f(z) = g(w(z))$.

Chebyshev polynomials, which is used by us in this paper, play a considerable act in numerical analysis. We know that the Chebyshev polynomials are four kinds. The most of books and research articles related to specific orthogonal polynomials of Chebyshev family, contain essentially results of Chebyshev polynomials of first and second kinds $T_n(x)$ and $U_n(x)$ and their numerous uses in different applications, see Doha [6] and Mason [15].

The well-known kinds of the Chebyshev polynomials are the first and second kinds. In the case of real variable x on $(-1, 1)$, the first and second kinds are defined by

$$T_n(x) = \cos n\theta,$$
$$U_n(x) = \frac{\sin(n+1)\theta}{\sin\theta}$$

where the subcript n denotes the polynomial degree and where $x = \cos\theta$. We consider the function

$$\Phi(z,t) = \frac{1}{1 - 2tz + z^2}.$$

We note that if $t = \cos\alpha$, $\alpha \in \left(\frac{-\pi}{3}, \frac{\pi}{3}\right)$, then for all $z \in \mathbb{U}$

$$\Phi(z,t) = \frac{1}{1 - 2tz + z^2} = 1 + \sum_{n=1}^{\infty} \frac{\sin(n+1)\alpha}{\sin\alpha} z^n = 1 + 2\cos\alpha z + (3\cos^2\alpha - \sin^2\alpha)z^2 + \cdots .$$

Thus, we write

$$\Phi(z,t) = 1 + U_1(t)z + U_2(t)z^2 + ... \qquad (z \in \mathbb{U}, t \in (-1,1))$$

where $U_{n-1} = \frac{\sin(n \arccos t)}{\sqrt{1-t^2}}$ for $n \in \mathbb{N}$, are the second kind of the Chebyshev polynomials. Also, it is known that

$$U_n(t) = 2tU_{n-1}(t) - U_{n-2}(t),$$

and

$$U_1(t) = 2t; \qquad U_2(t) = 4t^2 - 1, \qquad U_3(t) = 8t^3 - 4t, \cdots . \tag{1.3}$$

The Chebyshev polynomials $T_n(t), t \in [-1, 1]$, of the first kind have the generating function of the form

$$\sum_{n=0}^{\infty} T_n(t)z^n = \frac{1 - tz}{1 - 2tz + z^2} \qquad (z \in \mathbb{U}).$$

All the same, the Chebyshev polynomials of the first kind $T_n(t)$ and the second kind $U_n(t)$ are well connected by the following relationship

$$\frac{dT_n(t)}{dt} = nU_{n-1}(t),$$
$$T_n(t) = U_n(t) - tU_{n-1}(t),$$

$$2T_n(t) = U_n(t) - U_{n-2}(t).$$

Several authors have introduced and investigated subclasses of bi-univalent functions and obtained bounds for the initial coefficients (see [2, 3, 12, 16, 21, 22, 24]).

The study of operators plays an important role in the geometric function theory and its related fields.Many differential and integral operators can be written in terms of convolution of certain analytic functions. It is observed that this formalism brings an ease in further mathematical exploration and also helps to understand the geometric properties of such operators better. The convolution or Hadamard product of two functions $f, h \in \mathcal{A}$ is denoted by $f * h$ and is defined as

$$(f * h)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, \tag{1.4}$$

where $f(z)$ is given by (1.1) and $h(z) = z + \sum\limits_{n=2}^{\infty} b_n z^n$.

For complex parameters $\alpha_1, \ldots, \alpha_l$ and $\beta_1, \ldots, \beta_m$ $(\beta_j \neq 0, -1, \ldots; j = 1, 2, \ldots, m)$ the *generalized hypergeometric function* ${}_lF_m(z)$ is defined by

$$\begin{aligned} {}_lF_m(z) \equiv {}_lF_m(\alpha_1, \ldots \alpha_l; \beta_1, \ldots, \beta_m; z) \ &:= \ \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \ldots (\alpha_l)_n}{(\beta_1)_n \ldots (\beta_m)_n} \, \frac{z^n}{n!} \\ (l \leq m+1;\ l, m \in \mathbb{N}_0 \ &:= \ \mathbb{N} \cup \{0\}; z \in \mathbb{U}) \end{aligned} \tag{1.5}$$

where $\mathbb{N}$ denotes the set of all positive integers and $(a)_n$ is the Pochhammer symbol defined by

$$(a)_n = \begin{cases} 1, & n = 0 \\ a(a+1)(a+2)\ldots(a+n-1), & n \in \mathbb{N}. \end{cases} \tag{1.6}$$

For positive real values of $\alpha_1, \ldots, \alpha_l$ and $\beta_1, \ldots, \beta_m$ $(\beta_j \neq 0, -1, \ldots; j = 1, 2, \ldots, m)$, let

$$\mathcal{H}(\alpha_1, \ldots \alpha_l; \beta_1, \ldots, \beta_m) : \mathcal{S} \to \mathcal{S}$$

be a linear operator defined by

$$[(\mathcal{H}(\alpha_1, \ldots \alpha_l; \beta_1, \ldots, \beta_m))(f)](z) = z \ {}_lF_m(\alpha_1, \alpha_2, \ldots \alpha_l; \beta_1, \beta_2 \ldots, \beta_m; z) * f(z)$$

$$\mathcal{H}_m^l f(z) = z + \sum_{n=2}^{\infty} \Gamma_n \ a_n \ z^n \tag{1.7}$$

where

$$\Gamma_n = \frac{(\alpha_1)_{n-1} \ldots (\alpha_l)_{n-1}}{(\beta_1)_{n-1} \ldots (\beta_m)_{n-1}} \, \frac{1}{(n-1)!} \tag{1.8}$$

$\alpha_i > 0, (i = 1, 2, ...l), \beta_j > 0, (j = 1, 2, ...m), l \leq m+1;\ l, m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. With the same conditions ,we also define

$$\mathcal{H}_m^l g(w) = f^{-1}(w) = w - a_2 \Gamma_2 w^2 + (2a_2^2 - a_3)\Gamma_3 w^3 + \cdots . \tag{1.9}$$

Unless otherwise stated,throughout our study let $\mathcal{H}_m^l f(z)$ $\mathcal{H}_m^l g(w)$ and Γ_n are given by (1.7),(1.9)and (1.8) respectively.

For notational simplicity, we use a shorter notation $\mathcal{H}_m^l[\alpha_1]$ for $\mathcal{H}(\alpha_1, \ldots \alpha_l; \beta_1, \ldots, \beta_m)$ in the sequel. It follows from (1.7) that

$$\mathcal{H}_1^2[1]f(z) = f(z), \mathcal{H}_1^2[2]f(z) = zf'(z)$$

The linear operator $\mathcal{H}_m^l[\alpha_1]$ is called Dziok-Srivastava operator (see [9]). Further by using the Gaussian hypergeometric function given by (1.7), Hohlov [11] introduced a generalized convolution operator $H_{a,b,c}$ as $H_{a,b,c}f(z) = z\ {}_2F_1(a,b,c;z) * f(z)$, contains as special cases most of the known linear integral or differential operators. For the suitable choices of l, m in turn the operator $\mathcal{H}_m^l[\alpha_1]$ includes various operators as remarked below:

Remark 1.1. For $f \in \mathcal{A}$,

$$\mathcal{H}_1^2(a,1;c)f(z) = \mathcal{L}(a,c)f(z) = \left(z + \sum_{n=2}^{\infty} \frac{(a)_{n-1}}{(c)_{n-1}} z^n\right) * f(z) = z + \sum_{n=2}^{\infty} \frac{(a)_{n-1}}{(c)_{n-1}} a_n z^n$$

was considered by Carlson and Shaffer [5].

Remark 1.2. For $f \in \mathcal{A}$, and $\mathcal{H}_1^2(\delta+1,1;1)f(z) = \frac{z}{(1-z)^{\delta+1}} * f(z) = \mathcal{D}^\delta f(z), (\delta > -1)$ given by $\mathcal{D}^\delta f(z) = z + \sum_{n=2}^{\infty} \begin{pmatrix} \delta+n-1 \\ n-1 \end{pmatrix} a_n z^n$,was introduced by Ruscheweyh [20].

Remark 1.3. For $f \in \mathcal{A}$, and $\mathcal{H}_1^2(c+1,1;c+2)f(z) = \frac{c+1}{z^c}\int_0^z t^{c-1}f(t)dt = \mathcal{J}_c f(z)$ where $c > -1$. The operator $\mathcal{J}_c$ was introduced by Bernardi [4]. In particular, the operator $\mathcal{J}_1$ was studied earlier by Libera [13] and Livingston[14].

Remark 1.4. For $f \in \mathcal{A}, \mathcal{H}_1^2(2,1;2-\lambda)f(z) = \Gamma(2-\lambda)z^\lambda \mathcal{D}_z^\lambda f(z) = \Omega^\lambda f(z),\ \ \lambda \notin \mathbb{N} \setminus \{1\}$ called Owa-Srivastava operator[23] and Ω^λ is also called Srivastava-Owa fractional derivative operator, where $\mathcal{D}_z^\lambda f(z)$ denotes the fractional derivative of $f(z)$ of order λ, studied by Owa [18].

2. Bi-Univalent function classes $\mathcal{M}_\Sigma^{l,m}(\lambda, \Phi(z,t))$ and $\mathcal{F}_\Sigma^{l,m}(\beta, \Phi(z,t))$

Motivated by recent works of Altinkaya and Yalcin [1] (also see [9]) and recent study on bi-univalent functions involving hypergeometric functions [17], in this section, we introduce two new subclass of Σ associated with Chebyshev polynomials and obtain the initial Taylor coefficients $|a_2|$ and $|a_3|$ for the function classes by subordination.

Definition 2.1. For $0 \le \lambda \le 1$, and $t \in (-1,1)$ a function $f \in \Sigma$ of the form (1.1) is said to be in the class $\mathcal{M}_\Sigma^{l,m}(\lambda, \Phi(z,t))$ if the following subordination hold:

$$(1-\lambda)\frac{z(\mathcal{H}_m^l f(z))'}{\mathcal{H}_m^l f(z)} + \lambda \frac{[z(\mathcal{H}_m^l f(z))']'}{(\mathcal{H}_m^l f(z))'} \prec \Phi(z,t) \tag{2.1}$$

and

$$(1-\lambda)\frac{w(\mathcal{H}_m^l g(w))'}{\mathcal{H}_m^l g(w)} + \lambda \frac{[w(\mathcal{H}_m^l g(w))']'}{(\mathcal{H}_m^l g(w))'} \prec \Phi(w,t), \tag{2.2}$$

where $z, w \in \mathbb{U}$ and g is given by (1.2).

We note that by specializing the parameters λ and suitably fixing the values for l, m in Definition 2.1, we introduce (not have been studied sofar) the following new subclasses of Σ as listed below:

Remark 2.2. Suppose $f(z) \in \Sigma$ and $t \in (-1,1)$, then we denote

(1) $\mathcal{M}_\Sigma^{l,m}(0, \Phi(z,t)) \equiv \mathcal{S}_\Sigma^{l,m}(\Phi(z,t))$,

(2) $\mathcal{M}_{\Sigma}^{l,m}(1,\Phi(z,t)) \equiv \mathcal{K}_{\Sigma}^{l,m}(\Phi(z,t))$,

(3) $\mathcal{M}_{\Sigma}^{2,1}(\lambda,\Phi(z,t)) \equiv \mathcal{M}_{\Sigma}(\lambda,\Phi(z,t))$,

(4) $\mathcal{M}_{\Sigma}^{2,1}(0,\Phi(z,t)) = \mathcal{S}_{\Sigma}^{*}(\Phi(z,t))$
and
(5) $\mathcal{M}_{\Sigma}^{2,1}(1,\Phi(z,t)) = \mathcal{K}_{\Sigma}(\Phi(z,t))$.

Due to Frasin and Aouf [10] and Panigarhi and Murugusundaramoorthy [19] we define the following new subclass .

Definition 2.3. For $0 \leq \beta \leq 1$, and $t \in (-1,1)$ a function $f \in \Sigma$ of the form (1.1) is said to be in the class $\mathcal{F}_{\Sigma}^{k}(\beta,\Phi(z,t))$ if the following subordination hold:

$$(1-\beta)\frac{\mathcal{H}_m^l f(z)}{z} + \beta(\mathcal{H}_m^l f(z))' \prec \Phi(z,t) \tag{2.3}$$

and

$$(1-\beta)\frac{\mathcal{H}_m^l g(w)}{w} + \beta(\mathcal{H}_m^l g(w))' \prec \Phi(w,t) \tag{2.4}$$

where $z,w \in \mathbb{U}$, $g = f^{-1}$, is given by (1.2).

In the Definition 2.3, by specializing the parameters β and suitably fixing the values for l,m which yields (not have been studied sofar) the following new subclasses of Σ as listed below:

Remark 2.4. Suppose $f(z) \in \Sigma$ and $t \in (-1,1)$, then we denote
(1) $\mathcal{F}_{\Sigma}^{l,m}(0,\Phi(z,t)) \equiv \mathcal{R}_{\Sigma}^{l,m}(\Phi(z,t))$,
(2) $\mathcal{F}_{\Sigma}^{l,m}(1,\Phi(z,t)) \equiv \mathcal{H}_{\Sigma}^{l,m}(\Phi(z,t))$,
(3) $f \in \mathcal{F}_{\Sigma}^{2,1}(\beta,\Phi(z,t)) \equiv \mathcal{F}_{\Sigma}(\beta,\Phi(z,t))$,
and
(4) $\mathcal{F}_{\Sigma}^{2,1}(1,\Phi(z,t)) \equiv \mathcal{H}_{\Sigma}(\Phi(z,t))$,

In the following theorems we determine the initial Taylor coefficients $|a_2|$ and $|a_3|$ for the function classes $f \in \mathcal{M}_{\Sigma}^{l,m}(\lambda,\Phi(z,t))$ and $f \in \mathcal{F}_{\Sigma}^{l,m}(\beta,\Phi(z,t))$.

Theorem 2.5. *Let f given by* (1.1) *be in the class $\mathcal{M}_{\Sigma}^{l,m}(\lambda,\Phi(z,t))$ and $t \in (0,1)$. Then*

$$|a_2| \leq \frac{2t\sqrt{2t}}{\sqrt{|[2(1+2\lambda)\Gamma_3 - (\lambda^2+5\lambda+2)\Gamma_2^2]4t^2 + (1+\lambda)^2\Gamma_2^2|}} \tag{2.5}$$

and

$$|a_3| \leq \frac{4t^2}{(1+\lambda)^2\Gamma_2^2} + \frac{t}{(1+2\lambda)\Gamma_3} \tag{2.6}$$

where $0 \leq \lambda \leq 1$ and $t \neq \frac{1}{\sqrt{2}}$.

Proof. Let $f \in \mathcal{M}_{\Sigma}^{l,m}(\lambda,\Phi(z,t))$ and $g = f^{-1}$. From(2.1) and (2.2), we have

$$(1-\lambda)\frac{z(\mathcal{H}_m^l f(z))'}{\mathcal{H}_m^l f(z)} + \lambda\frac{[z(\mathcal{H}_m^l f(z))']'}{(\mathcal{H}_m^l f(z))'} = \Phi(z,t) \tag{2.7}$$

and

$$(1-\lambda)\frac{w(\mathcal{H}_m^l g(w))'}{\mathcal{H}_m^l g(w)} + \lambda\frac{[w(\mathcal{H}_m^l g(w))']'}{(\mathcal{H}_m^l g(w))'} = \Phi(w,t). \tag{2.8}$$

Define the functions $u(z)$ and $v(w)$ by

$$u(z) = c_1 z + c_2 z^2 + \cdots \tag{2.9}$$

and

$$v(w) = d_1 w + d_2 w^2 + \cdots \tag{2.10}$$

are analytic in $\mathbb{U}$ with $u(0) = 0 = v(0)$ and $|u(z)| < 1$, $|v(w)| < 1$, for all $z \in \mathbb{U}$. It is well-known that

$$|u(z)| = |c_1 z + c_2 z^2 + \cdots| < 1 \qquad \text{and} \qquad |v(w)| = |d_1 w + d_2 w^2 + \cdots| < 1, z, w \in \mathbb{U}, \tag{2.11}$$

then

$$|c_j| \leq 1 \qquad \text{and} \qquad |d_j| \leq 1 \quad \text{for all} \quad j \in \mathbb{N}. \tag{2.12}$$

Using (2.9) and (2.10) in (2.7) and (2.8) respectively, we have

$$(1-\lambda)\frac{z(\mathcal{H}_m^l f(z))'}{\mathcal{H}_m^l f(z)} + \lambda \frac{[z(\mathcal{H}_m^l f(z))']'}{(\mathcal{H}_m^l f(z))'} = 1 + U_1(t)u(z) + U_2(t)u^2(z) + \cdots, \tag{2.13}$$

and

$$(1-\lambda)\frac{w(\mathcal{H}_m^l g(w))'}{\mathcal{H}_m^l g(w)} + \lambda \frac{[w(\mathcal{H}_m^l g(w))']'}{(\mathcal{H}_m^l g(w))'} = 1 + U_1(t)v(w) + U_2(t)v^2(w) + \cdots. \tag{2.14}$$

In light of (1.1) - (1.9), and from (2.13) and (2.14), we have

$$\begin{aligned} 1 + (1+\lambda)\Gamma_2 a_2 z + [2(1+2\lambda)\Gamma_3 a_3 - (1+3\lambda)\Gamma_2^2 a_2^2] z^2 + \cdots \\ = 1 + U_1(t)c_1 z + [U_1(t)c_2 + U_2(t)c_1^2] z^2 + \cdots, \end{aligned}$$

and

$$\begin{aligned} 1 - (1+\lambda)\Gamma_2 a_2 w + \{[(8\lambda+4)\Gamma_3 - (3\lambda+1)\Gamma_2^2)] a_2^2 - 2(1+2\lambda)\Gamma_3 a_3\} w^2 + \cdots \\ = 1 + U_1(t)d_1 w + [U_1(t)d_2 + U_2(t)d_1^2] w^2 + \cdots. \end{aligned}$$

which yields the following relations:

$$\begin{aligned} (1+\lambda)\Gamma_2 a_2 &= U_1(t)c_1, & (2.15)\\ -(1+3\lambda)\Gamma_2^2 a_2^2 + 2(1+2\lambda)\Gamma_3 a_3 &= U_1(t)c_2 + U_2(t)c_1^2 & (2.16) \end{aligned}$$

and

$$\begin{aligned} -(1+\lambda)\Gamma_2 a_2 &= U_1(t)d_1, & (2.17)\\ (4(1+2\lambda)\Gamma_3 - (1+3\lambda)\Gamma_2^2)a_2^2 - 2(1+2\lambda)\Gamma_3 a_3 &= U_1(t)d_2 + U_2(t)d_1^2. & (2.18) \end{aligned}$$

From (2.15) and (2.17) it follows that

$$c_1 = -d_1 \tag{2.19}$$

and

$$2(1+\lambda)^2\Gamma_2^2 a_2^2 = U_1^2(t)(c_1^2 + d_1^2). \tag{2.20}$$

Adding (2.16) to (2.18) and using (2.20), we obtain

$$a_2^2 = \frac{U_1^3(t)(c_2 + d_2)}{2[\{2(1+2\lambda)\Gamma_3 - (1+3\lambda)\Gamma_2^2\}U_1^2(t) - (1+\lambda)^2\Gamma_2^2 U_2(t)]}.$$

Applying (2.12) to the coefficients c_2 and d_2, and using (1.3)we have

$$|a_2| \leq \frac{2t\sqrt{2t}}{\sqrt{|[2(1+2\lambda)\Gamma_3 - (\lambda^2+5\lambda+2)\Gamma_2^2]4t^2 + (1+\lambda)^2\Gamma_2^2|}}. \tag{2.21}$$

By subtracting (2.18) from (2.16) and using (2.19) and (2.20), we get

$$a_3 = \frac{U_1^2(t)(c_1^2+d_1^2)}{2(1+\lambda)^2\Gamma_2^2} + \frac{U_1(c_2-d_2)}{4(1+2\lambda)\Gamma_3}.$$

Using (1.3), once again applying (2.12) to the coefficients c_1, c_2, d_1 and d_2, we get

$$|a_3| \leq \frac{4t^2}{(1+\lambda)^2\Gamma_2^2} + \frac{t}{(1+2\lambda)\Gamma_3}. \tag{2.22}$$

□

By taking $\lambda = 0$ or $\lambda = 1$and $t \in (0,1)$, one can easily state the estimates $|a_2|$ and $|a_3|$ for the function classes $\mathcal{M}_\Sigma^{l,m}(0, \Phi(z,t)) = \mathcal{S}_\Sigma^{l,m}(\Phi(z,t))$ and $\mathcal{M}_\Sigma^{l,m}(1, \Phi(z,t)) = \mathcal{K}_\Sigma^{l,m}(\Phi(z,t))$ respectively.

Remark 2.6. Let f given by (1.1) be in the class $\mathcal{S}_\Sigma^{l,m}(\Phi(z,t))$. Then

$$|a_2| \leq \frac{2t\sqrt{2t}}{\sqrt{|[\Gamma_3 - \Gamma_2^2]8t^2 + \Gamma_2^2|}}$$

and

$$|a_3| \leq \frac{4t^2}{\Gamma_2^2} + \frac{t}{\Gamma_3}.$$

Remark 2.7. Let f given by (1.1) be in the class $\mathcal{K}_\Sigma^{l,m}(\Phi(z,t))$. Then

$$|a_2| \leq \frac{t\sqrt{2t}}{\sqrt{|[3\Gamma_3 - 4\Gamma_2^2]2t^2 + \Gamma_2^2|}} \tag{2.23}$$

and

$$|a_3| \leq \frac{t^2}{\Gamma_2^2} + \frac{t}{3\Gamma_3}. \tag{2.24}$$

For $l = 2, m = 1$, Theorem 2.5 yields the following corollary.

Corollary 2.8. *Let f given by* (1.1) *be in the class $\mathcal{M}_\Sigma(\lambda, \Phi(z,t))$. Then*

$$|a_2| \leq \frac{2t\sqrt{2t}}{\sqrt{|(1+\lambda)^2 - (\lambda^2+\lambda)4t^2|}}$$

and

$$|a_3| \leq \frac{4t^2}{(1+\lambda)^2} + \frac{t}{(1+2\lambda)}$$

where $0 \leq \lambda \leq 1$ *and* $t \neq \frac{1}{\sqrt{2}}$.

By taking $l = 2, m = 1$, in the above remarks we get the estimates $|a_2|$ and $|a_3|$ for the function classes $\mathcal{S}_\Sigma^*(\Phi(z,t))$ and $\mathcal{K}_\Sigma(\Phi(z,t))$.

Remark 2.9. Let f given by (1.1) be in the class $\mathcal{S}^*_\Sigma(\Phi(z,t))$. Then

$$|a_2| \leq 2t\sqrt{2t}$$

and

$$|a_3| \leq 4t^2 + t.$$

Remark 2.10. Let f given by (1.1) be in the class $\mathcal{K}_\Sigma(\Phi(z,t))$. Then for $t \neq \frac{1}{\sqrt{2}}$,

$$|a_2| \leq \frac{t\sqrt{2t}}{\sqrt{|1-2t^2|}}$$

and

$$|a_3| \leq t^2 + \frac{t}{3}$$

where $t \neq \frac{1}{\sqrt{2}}$.

Theorem 2.11. *Let f given by (1.1) be in the class $\mathcal{F}^{l,m}_\Sigma(\beta, \Phi(z,t))$ and $t \in (0,1)$. Then*

$$|a_2| \leq \frac{2t\sqrt{2t}}{\sqrt{|[(1+2\beta)\Gamma_3 - (1+\beta)^2\Gamma_2^2]4t^2 + (1+\beta)^2\Gamma_2^2|}} \tag{2.25}$$

and

$$|a_3| \leq \frac{4t^2}{(1+\beta)^2\Gamma_2^2} + \frac{2t}{(1+2\beta)\Gamma_3}. \tag{2.26}$$

Proof. Proceeding as in the proof of Theorem 2.5 we can arrive the following relations.

$$(1+\beta)\Gamma_2 a_2 = U_1(t)c_1 \tag{2.27}$$

$$(1+2\beta)\Gamma_3 a_3 = U_1(t)c_2 + U_2(t)c_1^2 \tag{2.28}$$

and

$$-(1+\beta)\Gamma_2 a_2 = U_1(t)d_1 \tag{2.29}$$

$$2(1+2\beta)\Gamma_3 a_2^2 - (1+2\beta)\Gamma_3 a_3 = U_1(t)d_2 + U_2(t)d_1^2. \tag{2.30}$$

From (2.27) and (2.29) it follows that

$$c_1 = -d_1 \tag{2.31}$$

and

$$2(1+\beta)^2\Gamma_2^2 a_2^2 = U_1^2(t)(c_1^2 + d_1^2). \tag{2.32}$$

From (2.28), (2.30) and (2.32), we obtain

$$a_2^2 = \frac{U_1^3(t)(c_2+d_2)}{2[(1+2\beta)\Gamma_3 U_1^2(t) - (1+\beta)^2\Gamma_2^2 U_2(t)]}.$$

Using (1.3) and (2.12) to the coefficients c_2 and d_2, we immediately get the desired estimate on $|a_2|$ as asserted in (2.25).

By subtracting (2.30) from (2.28) and using (2.31) and (2.32), we get

$$a_3 = \frac{U_1^2(t)(c_1^2+d_1^2)}{2(1+\beta)^2\Gamma_2^2} + \frac{U_1(t)(c_2-d_2)}{2(1+2\beta)\Gamma_3}.$$

Again using (1.3) and (2.12) to the coefficients c_1, c_2, d_1 and d_2, we get the desired estimate on $|a_3|$ as asserted in (2.26). □

Remark 2.12. Let f given by (1.1) be in the class $\mathcal{R}_{\Sigma}^{l,m}(\Phi(z,t))$. Then

$$|a_2| \leq \frac{2t\sqrt{2t}}{\sqrt{|(\Gamma_3 - \Gamma_2^2)4t^2 + \Gamma_2^2|}}$$

and

$$|a_3| \leq \frac{4t^2}{\Gamma_2^2} + \frac{2t}{\Gamma_3}.$$

Remark 2.13. Let f given by (1.1) be in the class $\mathcal{H}_{\Sigma}^{l,m}(\Phi(z,t))$. Then

$$|a_2| \leq \frac{t\sqrt{2t}}{\sqrt{|(3\Gamma_3 - 4\Gamma_2^2)t^2 + \Gamma_2^2|}}$$

and

$$|a_3| \leq \frac{t^2}{\Gamma_2^2} + \frac{2t}{3\Gamma_3}.$$

By taking $l = 2, m = 1$ we deduce the following results

Remark 2.14. Let f given by (1.1) be in the class $\mathcal{F}_{\Sigma}(\beta, \Phi(z,t))$. Then

$$|a_2| \leq \frac{2t\sqrt{2t}}{\sqrt{|(1+\beta)^2 - 4t^2\beta^2|}}$$

and

$$|a_3| \leq \frac{4t^2}{(1+\beta)^2} + \frac{2t}{(1+2\beta)}.$$

Remark 2.15. Let f given by (1.1) be in the class $\mathcal{F}_{\Sigma}^{2,1}(1, \Phi(z,t)) \equiv \mathcal{H}_{\Sigma}(\Phi(z,t))$. Then

$$|a_2| \leq \frac{t\sqrt{2t}}{\sqrt{|1-t^2|}}$$

and

$$|a_3| \leq t^2 + \frac{2t}{3}.$$

Remark 2.16. Let f given by (1.1) be in the class $\mathcal{F}_{\Sigma}^{2,1}(0, \Phi(z,t)) \equiv \mathcal{R}_{\Sigma}(\Phi(z,t))$. Then

$$|a_2| \leq 2t\sqrt{2t}$$

and

$$|a_3| \leq 4t^2 + 2t.$$

3. Fekete-Szegö inequalities

Due to Zaprawa [25], in this section we obtain the Fekete-Szegö inequality for the function classes $\mathcal{M}_{\Sigma}^{l,m}(\lambda, \Phi(z,t))$ and $\mathcal{F}_{\Sigma}^{l,m}(\beta, \Phi(z,t))$.

Theorem 3.1. *Let f given by* (1.1) *be in the class $\mathcal{M}_{\Sigma}^{l,m}(\lambda, \Phi(z,t))$ and $\mu \in \mathbb{R}$. Then we have*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{t}{(1+2\lambda)\Gamma_3}, & |\mu - 1| \leq \frac{\left|\frac{(1+\lambda)^2\Gamma_2^2}{4t^2} + 2(1+2\lambda)\Gamma_3 - (\lambda^2+5\lambda+2)\Gamma_2^2\right|}{2(1+2\lambda)\Gamma_3} \\ \frac{8|1-\mu|t^3}{|(2(1+2\lambda)\Gamma_3 - (\lambda^2+5\lambda+2)\Gamma_2^2)4t^2 + (1+\lambda)^2\Gamma_2^2|}, & |\mu - 1| \geq \frac{\left|\frac{(1+\lambda)^2\Gamma_2^2}{4t^2} + 2(1+2\lambda)\Gamma_3 - (\lambda^2+5\lambda+2)\Gamma_2^2\right|}{2(1+2\lambda)\Gamma_3} \end{cases}$$

Proof. From (2.16) and (2.18)

$$a_3-\mu a_2^2=(1-\mu)\frac{U_1^3(t)(c_2+d_2)}{(4(1+2\lambda)\Gamma_3-2(1+3\lambda)\Gamma_2^2)U_1^2(t)-2U_2(t)(1+\lambda)^2\Gamma_2^2}+\frac{U_1(t)(c_2-d_2)}{4(1+2\lambda)\Gamma_3} \tag{3.1}$$

$$=U_1(t)\left[\left(\Theta(\mu)+\frac{1}{4(1+2\lambda)\Gamma_3}\right)c_2+\left(\Theta(\mu)-\frac{1}{4(1+2\lambda)\Gamma_3}\right)d_2\right] \tag{3.2}$$

where

$$\Theta(\mu)=\frac{(1-\mu)U_1^2(t)}{2[2(1+2\lambda)\Gamma_3-(1+3\lambda)\Gamma_2^2)U_1^2(t)-(1+\lambda)^2\Gamma_2^2U_2(t)]}. \tag{3.3}$$

Then, in view of (1.3), we conclude that

$$|a_3-\mu a_2^2|\le\begin{cases}\frac{t}{(1+2\lambda)\Gamma_3}, & 0\le|\Theta(\mu)|\le\frac{1}{4(1+2\lambda)\Gamma_3}\\ 4t|\Theta(\mu)|, & |\Theta(\mu)|\ge\frac{1}{4(1+2\lambda)\Gamma_3}\end{cases}$$

Taking $\mu=1$, we have the following corollary.

Corollary 3.2. *If $f\in\mathcal{M}_\Sigma^{l,m}(\lambda,\Phi(z,t))$, then*

$$|a_3-a_2^2|\le\frac{t}{(1+2\lambda)\Gamma_3}. \tag{3.4}$$

Corollary 3.3. *Let f given by (1.1) be in the class $\mathcal{S}_\Sigma^{l,m}(\Phi(z,t))$ and $\mu\in\mathbb{R}$. Then we have*

$$|a_3-\mu a_2^2|\le\begin{cases}\frac{t}{\Gamma_3}, & |\mu-1|\le\frac{\left|\frac{\Gamma_2^2}{8t^2}+\Gamma_3-\Gamma_2^2\right|}{\Gamma_3}\\ \frac{8|1-\mu|t^3}{|((\Gamma_3-\Gamma_2^2)8t^2+\Gamma_2^2|}, & |\mu-1|\ge\frac{\left|\frac{\Gamma_2^2}{8t^2}+\Gamma_3-\Gamma_2^2\right|}{\Gamma_3}.\end{cases}$$

Especially, for $\mu=1$ if $f\in\mathcal{S}_\Sigma^{2,1}(\Phi(z,t))$ we obtain

$$|a_3-a_2^2|\le\ t. \tag{3.5}$$

Corollary 3.4. *Let f given by (1.1) be in the class $\mathcal{K}_\Sigma^{l,m}(\Phi(z,t))$ and $\mu\in\mathbb{R}$. Then we have*

$$|a_3-\mu a_2^2|\le\begin{cases}\frac{t}{3\Gamma_3}, & |\mu-1|\le\frac{\left|\frac{\Gamma_2^2}{2t^2}+3\Gamma_3-4\Gamma_2^2\right|}{3\Gamma_3}\\ \frac{2|1-\mu|t^3}{|((3\Gamma_3-4\Gamma_2^2)2t^2+\Gamma_2^2|}, & |\mu-1|\ge\frac{\left|\frac{\Gamma_2^2}{2t^2}+3\Gamma_3-\Gamma_2^2\right|}{3\Gamma_3}.\end{cases}$$

Especially, for $\mu=1$ if $f\in\mathcal{K}_\Sigma^{2,1}(\Phi(z,t))$ we obtain

$$|a_3-a_2^2|\le\ \frac{t}{3}. \tag{3.6}$$

Theorem 3.5. *Let f given by (1.1) be in the class $\mathcal{F}_\Sigma^{l,m}(\beta,\Phi(z,t))$ and $\mu\in\mathbb{R}$. Then we have*

$$|a_3-\mu a_2^2|\le\begin{cases}\frac{2t}{(1+2\beta)\Gamma_3}, & |\mu-1|\le\frac{\left|\frac{(1+\beta)^2\Gamma_2^2}{4t^2}+(1+2\beta)\Gamma_3-(1+\beta)^2\Gamma_2^2\right|}{(1+2\beta)\Gamma_3}\\ \frac{8|1-\mu|t^3}{|[(1+2\beta)\Gamma_3-(1+\beta)^2\Gamma_2^2]4t^2+(1+\beta)^2\Gamma_2^2|}, & |\mu-1|\ge\frac{\left|\frac{(1+\beta)^2\Gamma_2^2}{4t^2}+(1+2\beta)\Gamma_3-(1+\beta)^2\Gamma_2^2\right|}{(1+2\beta)\Gamma_3}\end{cases}$$

Proof. From (2.16) and (2.18)

$$a_3-\mu a_2^2=(1-\mu)\frac{U_1^3(t)(c_2+d_2)}{2[(1+2\beta)\Gamma_3U_1^2(t)-(1+\beta)^2\Gamma_2^2U_2(t)]}+\frac{U_1(t)(c_2-d_2)}{2(1+2\beta)\Gamma_3} \tag{3.7}$$

$$=U_1(t)\left[\left(\Upsilon(\mu)+\frac{1}{2(1+2\beta)\Gamma_3}\right)c_2+\left(\Upsilon(\mu)-\frac{1}{2(1+2\beta)\Gamma_3}\right)d_2\right] \tag{3.8}$$

where

$$\Upsilon(\mu)=\frac{(1-\mu)U_1^2(t)}{2[(1+2\beta)\Gamma_3U_1^2(t)-(1+\beta)^2\Gamma_2^2U_2(t)]}. \tag{3.9}$$

Then, in view of (1.3), we conclude that

$$|a_3-\mu a_2^2|\leq\begin{cases}\frac{2t}{(1+2\beta)\Gamma_3}, & 0\leq|\Upsilon(\mu)|\leq\frac{1}{2(1+2\beta)\Gamma_3}\\ 4t|\Upsilon(\mu)|, & |\Upsilon(\mu)|\geq\frac{1}{2(1+2\beta)\Gamma_3}\end{cases}$$

Taking $\mu=1$, we have the following corollary.

Corollary 3.6. *If $f\in\mathcal{F}_{\Sigma}^{l,m}(\beta,\Phi(z,t))$, then*

$$|a_3-a_2^2|\leq\frac{2t}{(1+2\beta)\Gamma_3}. \tag{3.10}$$

Corollary 3.7. *Let f given by (1.1) be in the class $\mathcal{R}_{\Sigma}^{l,m}(\Phi(z,t))$ and $\mu\in\mathbb{R}$. Then we have*

$$|a_3-\mu a_2^2|\leq\begin{cases}\frac{2t}{\Gamma_3}, & |\mu-1|\leq\frac{\left|\frac{\Gamma_2^2}{4t^2}+\Gamma_3-\Gamma_2^2\right|}{\Gamma_3}\\ \frac{8|1-\mu|t^3}{|[\Gamma_3-\Gamma_2^2]4t^2+\Gamma_2^2|}, & |\mu-1|\geq\frac{\left|\frac{\Gamma_2^2}{4t^2}+\Gamma_3-\Gamma_2^2\right|}{\Gamma_3}\end{cases}$$

Especially, for $\mu=1$ if $f\in\mathcal{R}_{\Sigma}^{2,1}(\Phi(z,t))$ we obtain

$$|a_3-a_2^2|\leq\ 2t. \tag{3.11}$$

Corollary 3.8. *Let f given by (1.1) be in the class $\mathcal{H}_{\Sigma}^{l,m}(\Phi(z,t))$ and $\mu\in\mathbb{R}$. Then we have*

$$|a_3-\mu a_2^2|\leq\begin{cases}\frac{2t}{3\Gamma_3}, & |\mu-1|\leq\frac{\left|\frac{\Gamma_2^2}{t^2}+3\Gamma_3-4\Gamma_2^2)\right|}{3\Gamma_3}\\ \frac{2|1-\mu|t^3}{|[3\Gamma_3-4\Gamma_2^2]t^2+\Gamma_2^2|}, & |\mu-1|\geq\frac{\left|\frac{\Gamma_2^2}{t^2}+3\Gamma_3-4\Gamma_2^2\right|}{3\Gamma_3}\end{cases}$$

Especially, for $\mu=1$ if $f\in\mathcal{H}_{\Sigma}^{2,1}(\Phi(z,t))$ we obtain

$$|a_3-a_2^2|\leq\ \frac{2t}{3}. \tag{3.12}$$

Concluding Remarks: Further Specializing the parameters l,m one can define the various other interesting subclasses of Σ involving the differential operators as stated in Remarks 1.1 to 1.4, and results (as in above Theorems)and the corresponding corollaries as mentioned above can be derived easily. The details involved may be left as an exercise for the interested reader.

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